

A Rigorous Bound on the Critical Exponent for the Number of Lattice Trees, Animals, and Polygons

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The number of n -site lattice trees (up to translation) is believed to behave asymptotically as $Cn^{-\theta}\lambda^n$, where θ is a critical exponent dependent only on the dimension d of the lattice. We present a rigorous proof that $\theta \geq (d-1)/d$ for any $d \geq 2$. The method also applies to lattice animals, site animals, and two-dimensional self-avoiding polygons. We also prove that $\theta \geq \nu$ when $d=2$, where ν is the exponent for the radius of gyration.

KEY WORDS: Critical exponent; lattice tree; lattice animal; self-avoiding polygon; subadditivity.

1. INTRODUCTION

In this paper we consider several kinds of lattice objects: trees, animals, site animals, and self-avoiding polygons. These objects will be defined precisely at the beginning of Section 2 (except for polygons, which are defined in Section 3).

These different objects have a few things in common. For definiteness, let \mathbf{Z}^d be the d -dimensional hypercubic lattice ($d \geq 2$) and let t_n to be number of n -site trees in \mathbf{Z}^d up to translation. Then it can be proven rigorously that there is a "growth constant" λ_t , with $1 < \lambda_t < \infty$, such that

$$\lim_{n \rightarrow \infty} t_n^{1/n} = \lambda_t = \sup_{n \geq 1} t_n^{1/n} \quad (1)$$

This follows from the supermultiplicative (or, after taking $-\log$, sub-additive) inequality

$$t_n t_m \leq t_{n+m} \quad \text{for } n, m \geq 1 \quad (2)$$

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Equation (2) is a consequence of the fact that any two trees may be (translated and) joined by a single bond to form a larger tree with the property that the two original trees lie on opposite sides of a hyperplane $x_1 = \text{const}$ (see Klein⁽⁹⁾ for details). The actual scaling behavior of t_n is believed to be

$$t_n \sim C_t n^{-\theta_t} \lambda_t^n \tag{3}$$

The same behavior is expected for other lattices; the constants C_t and λ_t should depend upon the lattice, but the critical exponent θ_t is believed to depend only on the dimension d . Observe that the second equality of (1) immediately implies that

$$\theta_t \geq 0 \tag{4}$$

assuming that the scaling behavior (3) indeed holds.

Everything in the preceding paragraph has a direct extension to animals and site animals.⁽⁸⁾ In our notation, we shall replace the t by a and s , respectively, so that the expected scaling behaviors are

$$a_n \sim C_a n^{-\theta_a} \lambda_a^n \quad \text{for animals} \tag{5}$$

$$s_n \sim C_s n^{-\theta_s} \lambda_s^n \quad \text{for site animals} \tag{6}$$

It is believed that trees, animals, and site animals are all in the same universality class⁽¹¹⁾ and in particular that $\theta_t = \theta_a = \theta_s$ in every dimension. Moreover, the exponent should depend only on the dimension and not on the actual lattice (as opposed to C_t and λ_t , for example). The values of the exponent θ are believed to be as given in the following table:

d	2	3	4	5	6	7	≥ 8
θ	1	3/2	1.83	2.07	2.25	2.39	5/2

The values for 2 and 3 dimensions come from the dimensional reduction calculation of Parisi and Sourlas⁽¹⁴⁾ and are believed to be exact. Mean-field behavior (with $\theta = 5/2$) is believed to occur for $d > 8$, with logarithmic corrections for $d = 8$. The values for 4, 5, 6, and 7 dimensions are derived from the exact (but nonrigorous) relation $\theta = 1 + (d - 2)\nu$,⁽¹⁴⁾ where ν is the exponent for the diameter of an average tree or animal, via the Flory-type approximation $\nu \approx 5/2(d + 2)$.^(6,1) These values for θ , as well as the relation $\theta_t = \theta_a = \theta_s$, have been supported numerically.⁽²⁾

For self-avoiding polygons, the situation is slightly complicated because of the concatenation procedure (see Section 3.2 of ref. 13; see also

Section 3 below). Nevertheless, if q_n is the number of n -step self-avoiding polygons, then we have

$$\lim_{n \rightarrow \infty} q_n^{1/n} = \lambda_q = \sup_{n \geq 1} [q_n / (d-1)]^{1/n} \tag{7}$$

and the conjectured scaling behavior is

$$q_n \sim C_q n^{-\theta_q} \lambda_q^n \tag{8}$$

For polygons, we implicitly restrict to even values of n everywhere, because there are no polygons of odd length in \mathbf{Z}^d . We note that the usual notation for self-avoiding polygons is μ instead of λ_q and $\alpha_{\text{sing}} - 3$ instead of $-\theta_q$. The hyperscaling relation $2 - \alpha_{\text{sing}} = dv$ (which has not been proven rigorously) leads to the following values:

$$\begin{array}{cccc} d & 2 & 3 & \geq 4 \\ \theta_q & 5/2 & 2.76\dots & 1 + d/2 \end{array}$$

See Madras and Slade⁽¹³⁾ for more discussion.

The main result of the present paper is a rigorous proof of the lower bound

$$\theta \geq \frac{d-1}{d} \tag{9}$$

for each of the models described above (except that we only consider $d=2$ for polygons). Since the critical exponent has not been rigorously proven to exist in general, we formulate the result more precisely in the following two theorems. (Precise definitions of all terms are given at the beginnings of the next two sections.)

Theorem 1.1. Let $d \geq 2$. For the lattice \mathbf{Z}^d , or more generally for any d -dimensional homogeneous lattice, we have the following bounds for the number of trees, animals, and site animals, respectively:

- (i) $t_n \leq \frac{1}{d2^{(d-1)/d}} n^{-(d-1)/d} \lambda_t^n$ for all $n \geq 1$
- (ii) $a_n \leq \frac{1}{d2^{(d-1)/d}} n^{-(d-1)/d} \lambda_a^n$ for all $n \geq 1$
- (iii) $s_n \leq \frac{1}{2^{(d-1)/d}} n^{-(d-1)/d} \lambda_s^{n+2}$ for all $n \geq 1$

Theorem 1.2. Let q_n be the number of self-avoiding polygons (up to translation) on the square lattice \mathbf{Z}^2 . Then there exists a finite constant C such that

$$q_n \leq C n^{-1/2} \lambda_q^n \quad \text{for all (even) } n \geq 1$$

A byproduct of the proof technique is the following bound, valid in two dimensions for any of the models consider above:

$$\theta \geq \nu$$

The critical exponent ν is the one governing the length scale of the tree (or animal or polygon). A common way to define ν is that the mean square radius of gyration of an n -site tree is asymptotically proportional to $n^{2\nu}$. [We say that $f(n)$ is *asymptotically proportional* to $g(n)$ if $f(n)/g(n)$ converges to a finite nonzero constant.] An alternative way is to say that the mean span is asymptotically proportional to n^ν , and this is the definition that we shall use (see Section 4 for the precise definitions).

Theorem 1.3. Let $d=2$ for any of the models considered in Theorems 1.1 or 1.2. If the mean span for the model is asymptotically proportional to n^ν , then there exists a (lattice-dependent) constant B_* such that

$$*_n \leq B_* n^{-\nu} \lambda_*^n \quad \text{for all } n \geq 1$$

(Here $*$ is one of $t, a, s,$ or $q,$ depending on the model under consideration.)

To conclude this section, we shall describe previous rigorous bounds on the exponent θ . As mentioned at the beginning of this section, sub-additivity implies that $\theta_t, \theta_a,$ and θ_s are all nonnegative, in the sense that $t_n \leq \lambda_t^n$ for all n , etc. Hara and Slade⁽⁵⁾ proved that (3) and (5) hold with $\theta_t = \theta_a = \frac{5}{2}$ for \mathbf{Z}^d when d is sufficiently large. They also proved this for any $d > 8$ on a “spread-out” lattice with sufficiently large range L . (A “spread-out” lattice of range L has the points of \mathbf{Z}^d as its sites, and two distinct sites (x_1, \dots, x_d) and (y_1, \dots, y_d) are joined by a bond whenever $|x_i - y_i| \leq L$ for every $i = 1, \dots, d$.) It has also been proven that θ_t and θ_a are bounded above by $\frac{5}{2}$ in every dimension, in the weaker sense that

$$\sum_{n=1}^{\infty} n^2 t_n z^n \geq \text{const.} (\lambda_t^{-1} - z)^{-1/2} \quad \text{as } z \nearrow \lambda_t^{-1},$$

and also with t replaced by a (Bovier, Fröhlich, and Glaus,⁽¹⁾ Tasaki and Hara;⁽¹⁵⁾ Hara and Slade⁽³⁾). This bound does not translate into a termwise lower bound for t_n or a_n . The only termwise lower bound known in general d is $C \exp[-\delta(\log n)^2] \lambda_t^n \leq t_n$ for some positive constants C and δ (Janse van Rensburg⁽⁷⁾).

For self-avoiding polygons, subadditivity implies $\theta_q \geq 0$. Using methods very different from the present paper, Madras⁽¹²⁾ proved that $\theta_q \geq \frac{1}{2}$ in $d=2$, but only at the level of generating functions: i.e.,

$$\sum_{n=1}^{\infty} q_n z^n \leq \text{const.} (\lambda_q^{-1} - z)^{-1/2} \quad \text{as } z \nearrow \lambda_q^{-1},$$

The same paper proved, in the same sense, that $\theta_q \geq 1$ in $d=3$ and $\theta_q > 1$ for $d \geq 4$. Hara and Slade⁽⁴⁾ proved that $\theta_q \geq 1 + d/2$ for $d \geq 5$, again at the level of generating functions. (But see also Theorem 6.1.3 of ref. 13.) Finally, in general d , there are termwise lower bounds on q_n that imply $\theta_q \leq d + 5$ if another critical exponent exists (see ref. 13, Section 8.1).

2. DEFINITIONS AND THE PROOF OF THEOREM 1.1

In this section, we define the basic objects of study and we give the complete proof of the main result for the case of trees in \mathbf{Z}^d .

We will work on periodic d -dimensional lattices \mathcal{L} with $d \geq 2$. We view \mathcal{L} as an infinite graph, consisting of *sites* and *bonds*, that have been embedded in d -dimensional Euclidean space \mathbf{R}^d in a periodic manner. We write $\langle x, y \rangle$ to denote the bond whose endpoints are the sites $x, y \in \mathbf{R}^d$. The edges are not directed (i.e., $\langle x, y \rangle = \langle y, x \rangle$). We always fix the origin 0 to be one of the sites of \mathcal{L} . An important special case is the simple hypercubic lattice \mathbf{Z}^d , whose sites are the points $(x_1, \dots, x_d) \in \mathbf{R}^d$ with integer coordinates, and whose bonds join pairs of sites that are unit distance apart.

In this section, we shall only consider lattices with the following property: there is a finite set $V \subset \mathbf{R}^d \setminus \{0\}$ such that $\langle x, y \rangle$ is a bond of \mathcal{L} if and only if $x - y \in V$. A lattice with this property is said to be *homogeneous*. Then \mathbf{Z}^d is homogeneous, as are the triangular, face-centered, and body-centered cubic lattices, as well as the “spread-out” lattices of Hara and Slade.⁽⁵⁾ We shall also assume that \mathcal{L} is truly d -dimensional, in the sense that V contains a basis of \mathbf{R}^d . In fact, by applying an invertible linear transformation to \mathbf{R}^d if necessary (which does not affect the number of animals, etc.), we can and shall assume that $e_i \in V$ for $i = 1, \dots, d$, where e_1, \dots, e_d is the standard orthonormal basis of \mathbf{R}^d .

An *animal* is a finite connected subgraph of the lattice \mathcal{L} (in some other papers, this is called a *bond animal*). A *tree* is an animal that has no cycles. A *site animal* is an animal with the property that if two sites of the animal are joined by a bond in the lattice, then that bond must belong to the animal.

If G is a subgraph of \mathcal{L} , and if x is a vector in \mathbf{R}^d , then we define $G + x$ to be the translation of G by x : i.e., the graph whose sites are

$\{z + x: z \text{ is a site of } G\}$, and whose bonds are $\{\langle y + x, z + x \rangle: \langle y, z \rangle \text{ is a bond of } G\}$.

For each integer $n \geq 1$, let \mathcal{A}_n be the set of all animals with n sites whose lexicographically smallest site is the origin. [We say that (x_1, \dots, x_d) is lexicographically smaller than (y_1, \dots, y_d) if there exists a j such that $x_i = y_i$ for all $i = 1, \dots, j - 1$ and $x_j < y_j$.] For a homogeneous lattice, if τ is any animal with n sites, then there is a unique $x \in \mathbf{R}^d$ such that $\tau - x \in \mathcal{A}_n$ (in fact, x is the lexicographically smallest site of τ). Using $|\cdot|$ to denote cardinality, we define

$$a_n := |\mathcal{A}_n| \quad (n \geq 1)$$

This is “the number of animals with n sites, up to translation.” Similarly, we let \mathcal{T}_n (respectively, \mathcal{S}_n) be the set of all trees (respectively, site animals) with n sites whose lexicographically smallest site is the origin. We also define

$$t_n := |\mathcal{T}_n| \quad \text{and} \quad s_n := |\mathcal{S}_n| \quad (n \geq 1)$$

as the number of trees and site animals, respectively, with n sites, up to translation.

For $x = (x_1, \dots, x_d) \in \mathbf{R}^d$ and $i \in \{1, \dots, d\}$, let

$$\text{proj}_i(x) = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_d)$$

be the projection of x onto the subspace of \mathbf{R}^d orthogonal to the x_i axis. Also, for an animal τ , let

$$\text{Proj}_i(\tau) = \{\text{proj}_i(x): x \text{ is a site of } \tau\}$$

Observe that if τ has n sites, then $|\text{Proj}_i(\tau)| \leq n$. For example, if rod is the line segment from 0 to $(n - 1, 0, \dots)$, then $|\text{Proj}_1(rod)| = 1$, while $|\text{Proj}_i(rod)| = n$ for all $i \neq 1$. Also note that if $n^{1/d}$ is an integer, and if the sites of τ exactly fill a hypercube of side $n^{1/d}$, then $|\text{Proj}_i(\tau)| = n^{(d-1)/d}$ for every i . The following inequality of Loomis and Whitney⁽¹⁰⁾ will be used below.

Theorem 2.1. Let τ be a graph with n sites in a d -dimensional lattice. Then

$$\prod_{i=1}^d |\text{Proj}_i(\tau)| \geq n^{d-1}$$

Note that the theorem is simple to prove in $d = 2$, but much subtler for $d > 2$. One immediate consequence of this theorem and the arithmetic-geometric mean inequality is that

$$\frac{1}{d} \sum_{i=1}^d |\text{Proj}_i(\tau)| \geq n^{(d-1)/d} \tag{10}$$

whenever τ has n sites. Intuitively, this says that the average projection of a graph is at least as large as the average projection of a hypercube with the same number of sites.

The proof of Theorem 1.1 for trees and animals is an immediate consequence of the following two lemmas and Eq. (1).

Lemma 2.2. Let \mathcal{L} be a homogeneous d -dimensional lattice. Then for every $n \geq 1$,

$$t_{2n} \geq dn^{(d-1)/d} t_n^2 \quad \text{and} \quad a_{2n} \geq dn^{(d-1)/d} a_n^2 \tag{11}$$

Lemma 2.3. Let λ be a positive number and let v_1, v_2, \dots be a positive sequence such that

$$\lim_{n \rightarrow \infty} v_n^{1/n} = \lambda \tag{12}$$

Also assume that there are numbers $B > 0$ and $p \geq 0$, and an integer $k \geq 0$, such that

$$v_{2n+k} \geq Bn^p v_n^2 \quad \text{for every } n \geq 1 \tag{13}$$

Then

$$v_n \leq \frac{1}{B2^p n^p} \lambda^{n+k} \quad \text{for every } n \geq 1 \tag{14}$$

Before we prove these two lemmas, we record the following notation and elementary lemma. If b is a bond of a graph G , then $G \setminus b$ is the subgraph of G obtained by removing b from the set of bonds.

Lemma 2.4. Let G be a connected graph with $2n$ sites. Then there exists *at most* one bond b such that $G \setminus b$ is a disconnected graph with two connected components of exactly n sites each.

Proof. Suppose that there were two such bonds, $b^{(1)}$ and $b^{(2)}$. Let A and B be the two connected components of $G \setminus b^{(1)}$. Without loss of generality, assume that $b^{(2)}$ belongs to A . Let x be the endpoint of $b^{(1)}$ that belongs to A . Then one component of $G \setminus b^{(2)}$ contains all of B as well as x , which is more than n sites. This is a contradiction. ■

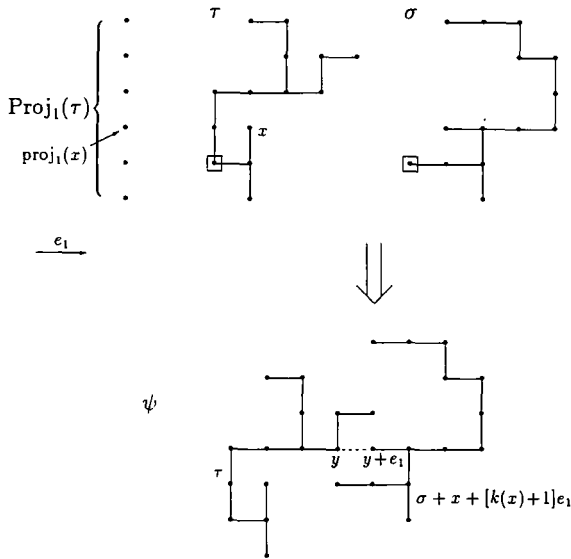


Fig. 1. The proof of Lemma 2.2 in the square lattice. Here $i=1$. The lexicographically smallest sites of τ and σ are enclosed by boxes. The dashed line in ψ is the added bond $\langle y, y + e_1 \rangle$.

Proof of Lemma 2.2. Here is the basic idea behind the proof in two dimensions (Fig. 1). Let τ and σ be two n -site animals (or trees) in \mathbb{Z}^2 . Translate σ vertically so that its lexicographically smallest site is on a horizontal line that intersects τ . There are at least $n^{1/2}$ choices for this line (if not, then replace τ by its rotation by $\pi/2$). Now we will just translate σ horizontally. Starting with σ far to the right of τ , move it in the $-e_1$ direction until just before it touches τ . In this position, add a bond to join σ to τ to get an animal (or tree) ψ with $2n$ sites. By Lemma 2.4, we can determine τ , σ , and the initial vertical translation unambiguously from ψ . This counting argument gives the desired result for $d=2$. The same idea works for $d \geq 3$, except that the initial “vertical” translation of σ becomes a translation of σ so that its lexicographically smallest site lies on a “horizontal” line (in the e_1 direction, say) that intersects τ ; there is one such line for every point of $\text{Proj}_1(\tau)$. By the Loomis–Whitney theorem, there are at least $n^{(d-1)/d}$ such lines (if not for τ , then for some rotation of τ).

We now give the full proof for trees; the proof for animals is virtually identical. Fix $n \geq 1$, fix $i \in \{1, \dots, d\}$, and fix two trees τ and σ in \mathcal{T}_n . Recall that e_i is the unit vector in the positive x_i direction. For each site x of τ , let

$$k(x) = \max\{k \in \mathbf{R} : (\sigma + x + ke_i) \cap \tau \neq \emptyset\}$$

(The above intersection refers to the set of sites that are in both $\sigma + x + ke_i$ and τ .) Observe that $(\sigma + x + ke_i) \cap \tau$ is empty for all sufficiently large k , while it contains x for $k=0$. Therefore $k(x)$ is a well-defined finite nonnegative number. Also observe that if x and x' are two sites of τ with $\text{proj}_i(x) = \text{proj}_i(x')$, then $x + k(x)e_i = x' + k(x')e_i$. Choose a site $y \in (\sigma + x + k(x)e_i) \cap \tau$ (if there is more than one site in the intersection, then choose the lexicographically smallest site). Let ψ be the animal consisting of the union of τ , $(\sigma + x + [k(x) + 1]e_i)$, and the bond $\langle y, y + e_i \rangle$.

We need to show that ψ , as obtained in the above paragraph, is in \mathcal{F}_{2n} . It is evident that ψ is a tree with $2n$ sites, so we only need to show that 0 is the lexicographically smallest site of ψ . We know that 0 is a site of ψ (since it is a site of τ), and that it is lexicographically smaller than all of the sites of τ . Suppose that z is a site of ψ that is not in τ . Then z can be written $y + x + [k(x) + 1]e_i$ for some site y of σ . We know that 0 is lexicographically smaller than each of y , x , and $[k(x) + 1]e_i$ [recall $k(x) \geq 0$], and so 0 must be lexicographically smaller than their sum. Therefore 0 is lexicographically smaller than z . We conclude that ψ is in \mathcal{F}_{2n} .

Now suppose that we are given only the final tree ψ . We claim that we can determine τ , σ , i , and $\text{proj}_i(x)$ unambiguously. By Lemma 2.4, there is a unique bond b such that $\psi \setminus b$ consists of two components of n sites each. Let u be the lexicographically smaller of the two endpoints of b , and let v be the other endpoint. Then $v - u = e_i$, so i is determined. Let ψ_u (respectively, ψ_v) be the component of $\psi \setminus b$ that contains u (respectively, v). Then $\psi_u = \tau$ and $\psi_v = (\sigma + x + [k(x) + 1]e_i)$. So τ is determined. Let z be the lexicographically smallest point of ψ_v ; then $z = x + [k(x) + 1]e_i$. This allows us to determine both $\sigma (= \psi_v - z)$ and $\text{proj}_i(x) [= \text{proj}_i(z)]$. Thus our claim has been proven.

Now consider the set of all possible ψ 's in \mathcal{F}_{2n} that can be formed by the procedure of the first paragraph. By the claim of the third paragraph,

$$\begin{aligned} \text{number of } \psi\text{'s} &= \sum_{i=1}^d \sum_{\tau, \sigma \in \mathcal{F}_n} |\text{Proj}_i(\tau)| \\ &= \sum_{i=1}^d \sum_{\tau \in \mathcal{F}_n} |\text{Proj}_i(\tau)| t_n \\ &\geq \sum_{\tau \in \mathcal{F}_n} dn^{(d-1)/d} t_n \quad [\text{by (10)}] \\ &= dn^{(d-1)/d} t_n^2 \end{aligned} \tag{15}$$

Since the number of ψ 's is at most t_{2n} , this proves the lemma. ■

Proof of Lemma 2.3. Let $u_m = B2^p(m - k)^p v_{m-k}$ for all $m > k$. Then

$$\lim_{m \rightarrow \infty} u_m^{1/m} = \lambda \tag{16}$$

by assumption (12). Also, by assumption (13),

$$\begin{aligned} u_{2m} &= B2^p(2m - k)^p v_{2m-k} \\ &\geq B2^p(2m - 2k)^p v_{2(m-k)+k} \\ &\geq B2^{2p}(m - k)^p [B(m - k)^p v_{m-k}^2] \\ &= u_m^2 \end{aligned}$$

for all $m > k$, and so

$$u_{2m}^{1/2m} \geq u_m^{1/m} \quad \text{for all } m > k \tag{17}$$

Now fix $n \geq 1$ and let

$$\tilde{u}_j = (u_{2^j(n+k)})^{1/2^j(n+k)} \quad \text{for } j = 0, 1, \dots$$

Equation (17) tells us that \tilde{u}_j is increasing in j , and its limit is λ [by (16)], so $\tilde{u}_j \leq \lambda$ for every j . In particular we have $\tilde{u}_0 \leq \lambda$, which says that

$$(B2^p n^p v_n)^{1/(n+k)} \leq \lambda$$

Since n is arbitrary, this proves the lemma. ■

The proof of the main result for site animals uses the following result instead of Lemma 2.2.

Lemma 2.5. Let \mathcal{L} be a homogeneous d -dimensional lattice. Then

$$s_{2n+2} \geq n^{-(d-1)/d} s_n^2 \quad \text{for all } n \geq 1$$

Remark. The proof is similar to that of Lemma 2.2, but not identical. This is because there may be many bonds of \mathcal{L} that join a site in τ to a site in $\sigma + x + [k(x) + 1]e_i$, and the definition of site animal requires that they all be in ψ . This prevents us from applying Lemma 2.4.

Proof of Lemma 2.5. Fix $n \geq 1$, fix $i \in \{1, \dots, d\}$, and fix two site animals τ and σ in \mathcal{L}_n . For each site x of τ , let

$$K(x) = \max \{k \in \mathbf{R}: \text{some site of } \sigma + x + ke_i \text{ is adjacent to some site of } \tau\}$$

Then $K(x)$ is a well-defined finite nonnegative number [in fact, $K(x) \geq 1$ because $\sigma + x + e_i$ contains $x + e_i$, which is adjacent to x , which is in τ]. Also observe that if x and x' are two sites of τ with $\text{proj}_i(x) = \text{proj}_i(x')$, then $x + K(x)e_i = x' + K(x')e_i$. Choose a site $y \in (\sigma + x + K(x)e_i)$ and a

site $z \in \tau$ such that y is adjacent to z (using lexicographic ordering to choose, if there is more than one possibility). Let ψ be the site animal consisting of the union of $\tau, z + e_i, y + e_i, (\sigma + x + [K(x) + 2] e_i)$, together with all those bonds that have both endpoints in ψ .

We claim that the ψ formed in the preceding paragraph is in \mathcal{S}_{2n+2} . Most of this can be shown by an argument just like the second paragraph of the proof of Lemma 2.2; the only new part is to show that ψ has $2n + 2$ sites. By definition of $K(x)$, we know that τ and $\sigma + x + [K(x) + 2] e_i$ have no sites in common, so it only remains to check that neither of them contains $z + e_i$ or $y + e_i$. First, notice that $z + e_i$ is adjacent to a site of τ (namely, z), and so the definition of $K(x)$ tells us that $z + e_i$ is not a site of $\sigma + x + [K(x) + 2] e_i$. Second, $z + e_i$ is adjacent to a site of $\sigma + x + [K(x) + 1] e_i$ (namely, $y + e_i$), and so the definition of $K(x)$ tells us that $z + e_i$ is not a site of τ . Therefore $z + e_i$ is not in $\tau \cup (\sigma + x + [K(x) + 2] e_i)$. A very similar argument shows that $y + e_i$ is not in $\tau \cup (\sigma + x + [K(x) + 2] e_i)$ either. Thus we have shown that ψ is indeed in \mathcal{S}_{2n+2} .

Let b be the bond $\langle y + e_i, z + e_i \rangle$. Then b is a bond of ψ , but we need to show that $\psi \setminus b$ is a disconnected graph. Let

$$G_1 = \{x: x \text{ is a site of } \tau\} \cup \{z + e_i\}$$

$$G_2 = \{x: x \text{ is a site of } \sigma + x + [K(x) + 2] e_i\} \cup \{y + e_i\}$$

We need to show that b is the only bond of the lattice that has one endpoint in G_1 and the other in G_2 . By the definition of $K(x)$, we know that there is no bond with one endpoint in τ and the other in $\sigma + x + [K(x) + 2] e_i$. Also $y + e_i$ is not adjacent to any site of τ , because $y + e_i$ is a site of $\sigma + x + [K(x) + 1] e_i$. Finally, $z + e_i$ is not adjacent to any site of $\sigma + x + [K(x) + 2] e_i$, because we know that z is not adjacent to any site of $\sigma + x + [K(x) + 1] e_i$. [We use the definition of $K(x)$ in these last two sentences.] Therefore $\psi \setminus b$ is indeed a disconnected graph, and its two components have G_1 and G_2 as their respective sets of sites.

Now suppose that we are given only the final site animal ψ . By Lemma 2.4, we can determine b, G_1 , and G_2 unambiguously (using the notation of the preceding paragraph). Knowing b tells us $z + e_i$ and $y + e_i$, and their deletion from G_1 and G_2 , respectively, determines τ and $\sigma + x + [K(x) + 2] e_i$. It may not be possible to determine i (since $z + e_i$ may be connected to τ by many bonds), but if we know the value of i , then we can determine $\text{proj}_i(x)$, and hence σ , as in the proof of Lemma 2.2. The rest of the present proof is very similar to the proof of Lemma 2.2, except that being unable to determine i from ψ gives us the equation

$$(\text{number of } \psi\text{'s obtained when } i = i') = \sum_{\tau, \sigma \in \mathcal{S}_n} |\text{Proj}_i(\tau)|$$

for each $i' = 1, \dots, d$, and hence the analog of (15) gives

$$\begin{aligned} \text{number of } \psi\text{'s} &\geq \frac{1}{d} \sum_{i'=1}^d (\text{number of } \psi\text{'s obtained when } i = i') \\ &\geq n^{(d-1)/d} s_n^2 \end{aligned}$$

Since the number of ψ 's is at most s_{2n+2} , this proves the lemma.

Proof of Theorem 1.1. As noted above, parts (i) and (ii) are immediate consequences of Lemmas 2.2 and 2.3 with $k=0$. Part (iii) is a consequence of Lemmas 2.5 and 2.3 with $k=2$. ■

3. SELF-AVOIDING POLYGONS

In this section we shall only consider the square lattice \mathbf{Z}^2 . It may well be possible to extend the results to other lattices, but this would require (even more) unpleasant details. It may also be possible to apply the method to self-avoiding polygons in more than two dimensions, but in light of the known rigorous bound $\theta_q \geq 1$ for $d \geq 3$,⁽¹²⁾ this hardly seems worth the effort, even though the result would be a termwise bound rather than a generating function bound.

An *n-step self-avoiding polygon* (or *n-step polygon*) is a lattice animal having n sites which consists of a single cycle; in other words, each site of the animal is the endpoint of exactly two bonds in the animal. Let \mathcal{Q}_n be the set of n -step polygons whose lexicographically smallest site is the origin. Then we define the number of n -step polygons (up to translation) to be

$$q_n := |\mathcal{Q}_n| \quad (n \geq 1)$$

Notice that $q_n = 0$ unless n is even and greater than 2.

The proof of Theorem 1.2 is similar in structure to the proofs given in the previous section. We begin with a discussion of the concatenation of two polygons.

A *plaquette* is a set of four bonds of \mathbf{Z}^2 corresponding to a unit square of the lattice. Formally, for each $x \in \mathbf{Z}^2$, let $P(x)$ be the set of bonds that have both endpoints in the set $\{x, x + e_1, x + e_2, x + e_1 + e_2\}$. Then the plaquettes are precisely the $P(x)$'s.

Suppose that π is an n -step polygon and ρ is an m -step polygon such that π and ρ have no sites in common. Suppose also that there are bonds b_π and b_ρ (belonging to π and ρ , respectively) which are both contained in the same plaquette $P = P(x)$. (Note that b_π and b_ρ must be parallel, since π and ρ are disjoint.) Let ψ be the graph consisting of $\pi \setminus b_\pi, \rho \setminus b_\rho$, and the two bonds of $P \setminus \{b_\pi, b_\rho\}$ (see Fig. 2). Then ψ is an $(n + m)$ -step polygon.

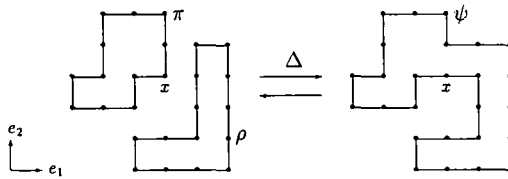


Fig. 2. The action of the symmetric difference operator: $\psi = (\pi \cup \rho) \Delta P(x)$ and $\pi \cup \rho = \psi \Delta P(x)$.

We shall represent this operation using the “symmetric difference” Δ as follows:

$$\psi = (\pi \cup \rho) \Delta P$$

Next, suppose that ψ is an M -step polygon, and that there is a plaquette P which contains exactly two bonds b_1 and b_2 of ψ and that b_1 and b_2 do not have any common endpoints. Let ϕ be the graph consisting of $\psi \setminus \{b_1, b_2\}$ and the two bonds of $P \setminus \{b_1, b_2\}$. Then ϕ is a graph consisting of two components, each a self-avoiding polygon, with total number of steps equal to M . Again using the symmetric difference notation, we represent this operation as follows:

$$\phi = \psi \Delta P$$

Thus, if ψ is as above, then we have $\psi = (\psi \Delta P) \Delta P$.

We can now state an analog of Lemma 2.4 for polygons. We omit the proof, which is very similar to that of Lemma 2.4.

Lemma 3.1. Let ψ be a $2n$ -step polygon in \mathbb{Z}^2 . Then there exists at most one plaquette P such that $\psi \Delta P$ has two components which are each n -step polygons.

Next we present the analog of Lemmas 2.2 and 2.5.

Lemma 3.2 (For $\mathcal{L} = \mathbb{Z}^2$). There exists a constant D such that

$$q_{2n+16} \geq Dn^{1/2} q_n^2 \quad \text{for all even } n \geq 4$$

Proof. Fix an even $n \geq 4$ and fix two polygons τ and σ in \mathcal{Q}_n . Also fix a site x of τ .

We would like to concatenate τ and $\sigma + x + ke_1$ for some k , but there may not be a k and a plaquette P such that these two polygons are disjoint and each contains a bond of P . Therefore, we shall modify the original polygons before concatenation. To do this, we need to ensure that there is space in which to make the modifications.

For each $y \in \mathbb{Z}^2$, let $N(y)$ denote the three-point set $\{y, y + e_2, y - e_2\}$. For each site x of τ , let

$$K(x) = \max\{k \in \mathbb{Z} : \text{there exists a } y \in \mathbb{Z}^2 \text{ such that } \tau \cap N(y) \neq \emptyset \text{ and } (\sigma + x + ke_1) \cap N(y) \neq \emptyset\}$$

Then $K(x)$ is a well-defined finite nonnegative number (since we can take $y = x$ when $k = 0$). Also observe that if x and x' are two sites of τ with $\text{proj}_1(x) = \text{proj}_1(x')$, then $x + K(x)e_1 = x' + K(x')e_1$. Let $Y = (Y_1, Y_2)$ be the lexicographically smallest site of \mathbb{Z}^2 such that $\tau \cap N(Y) \neq \emptyset$ and $(\sigma + x + K(x)e_1) \cap N(Y) \neq \emptyset$.

Notice that for any m we have $(\sigma + x + [K(x) + m]e_1) \cap N(Y + me_1) \neq \emptyset$. Therefore the definition of $K(x)$ tells us that

$$\tau \cap N(Y + me_1) = \emptyset \quad \text{for all } m \geq 1 \tag{18}$$

Also, since $(\sigma + x + [K(x) + m]e_1) \cap N(Y) = \emptyset$ for all $m \geq 1$, we have

$$(\sigma + x + K(x)e_1) \cap N(Y - me_1) = \emptyset \quad \text{for all } m \geq 1 \tag{19}$$

The two assertions (18) and (19) tell us that we have some space near Y in which we can make some modifications to our polygons. (We will have to move them a bit farther apart after the modifications before concatenating them.)

We now give a procedure for modifying τ to get a new polygon π with $n + 8$ steps (modifications to σ will be done by the same procedure with a left-right reflection). First we define the following four polygons (see Fig. 3): β_{10} is the 10-step polygon consisting of the closed circuit from $(0, 0)$ to $(0, 1)$ to $(3, 1)$ to $(3, -1)$ to $(2, -1)$ to $(2, 0)$ to $(0, 0)$; β_{12} is the 12-step polygon consisting of the closed circuit from $(0, 0)$ to $(0, 1)$ to $(4, 1)$ to $(4, -1)$ to $(3, -1)$ to $(3, 0)$ to $(0, 0)$; and $\hat{\beta}_{10}$ (respectively, $\hat{\beta}_{12}$) is the reflection of β_{10} (respectively, β_{12}) through the line $x_2 = 0$.

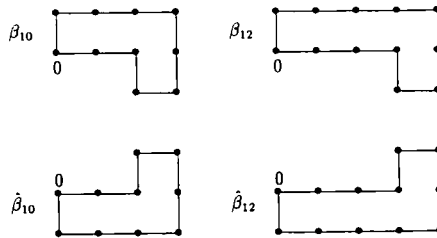


Fig. 3. The four polygons β_{10} , β_{12} , $\hat{\beta}_{10}$, and $\hat{\beta}_{12}$. The origin is labeled 0.

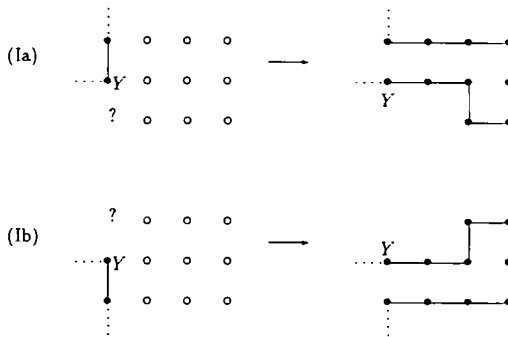


Fig. 4. Cases Ia and Ib. Empty sites are denoted by \circ ; sites that may or may not be empty are denoted by $?$; dotted lines represent bonds that may be in the polygon.

The procedure is given by the following case-by-case analysis:

Case I. Y is a site of τ . Then at least one of $\langle Y, Y+e_2 \rangle$ or $\langle Y, Y-e_2 \rangle$ must be in τ .

(Ia) $\langle Y, Y+e_2 \rangle$ is a bond of τ : Let π consist of $\tau \setminus \langle Y, Y+e_2 \rangle$ together with $(\beta_{10} + Y) \setminus \langle Y, Y+e_2 \rangle$ (see Fig. 4). Notice that (18) guarantees that none of the sites of $\beta_{10} + Y$ are in τ , except for Y and $Y+e_2$; hence, π is indeed a self-avoiding polygon.

(Ib) $\langle Y, Y-e_2 \rangle$ is a bond of τ : Let π consist of $\tau \setminus \langle Y, Y-e_2 \rangle$ together with $(\hat{\beta}_{10} + Y) \setminus \langle Y, Y-e_2 \rangle$ (see Fig. 4). Again, (18) guarantees that π is a self-avoiding polygon (we shall henceforth omit this comment).

Case II. Y is not a site of τ , but $Y+e_2$ is. Since $Y+e_2+e_1$ is not in τ either [by (18)], we deduce that $\langle Y+e_2, Y+e_2-e_1 \rangle$ must be a bond of τ . Call this bond b' . (See Fig. 5.)

(IIa) $Y-e_1$ is not in τ : Let π consist of $\tau \setminus b'$ together with $(\beta_{10} + Y - e_1) \setminus b'$ (see Fig. 6).

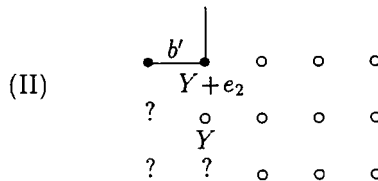


Fig. 5. The general configuration for case II. Empty sites are denoted by \circ ; sites that may or may not be empty are denoted by $?$.

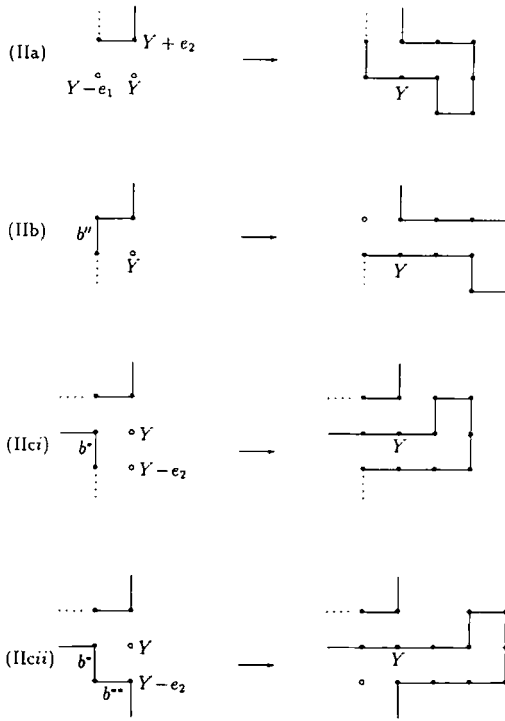


Fig. 6. Subcases of case II. Empty sites are denoted by \circ .

(IIb) $Y - e_1$ is in τ and the bond $b'' := \langle Y - e_1, Y + e_2 - e_1 \rangle$ is also in τ : Let π consist of $\tau \setminus \{b', b'', Y + e_2 - e_1\}$ together with $(\hat{\beta}_{12} + Y - e_1) \setminus \{b', b'', Y + e_2 - e_1\}$.

(IIc) $Y - e_1$ is in τ but the bond b'' is not in τ : Since Y is not in τ , we know that the bond $b^* := \langle Y - e_1, Y - e_1 - e_2 \rangle$ must be in τ . There are now two final subcases to consider, according as to whether $Y - e_2$ is in τ or not.

(IIci) $Y - e_2$ is not in τ : Let π consist of $\tau \setminus b^*$ together with $(\hat{\beta}_{10} + Y - e_1) \setminus b^*$.

(IIcii) $Y - e_2$ is in τ : Since $Y - e_2 + e_1$ is not in τ [by (18)], we deduce that the bond $b^{**} := \langle Y - e_2 - e_1, Y - e_2 \rangle$ is in τ . Therefore let π consist of $\tau \setminus \{b^*, b^{**}, Y - e_2 - e_1\}$ together with $(\hat{\beta}_{12} + Y - e_1) \setminus \{b^*, b^{**}, Y - e_2 - e_1\}$.

Case III. Y is not a site of τ , but $Y - e_2$ is. This case may be handled exactly as in case II, but with reflection through $x_2 = Y_2$, the horizontal line containing Y (so that β is replaced by $\hat{\beta}$, and so on).

Next modify $\sigma + x + K(x)e_1$ using the obvious analog of the above procedure (in which Figs. 4–6 have been reflected through the vertical line $x_1 = Y_1$) to get an $(n + 8)$ -step polygon ρ . Now polygons π and ρ will have some sites in common, so we want to translate ρ by e_1 just enough so that it no longer intersects π ; then we will concatenate. The next paragraph formalizes this operation.

Let w_1 be the largest integer such that (w_1, Y_2) is a site of π , and let u_1 be the smallest integer such that (u_1, Y_2) is a site of ρ . Let $\rho' = \rho + (w_1 - u_1 + 1)e_1$, and let $w = (w_1, Y_2)$. Then π and ρ' are disjoint [recall the definition of $K(x)$], while $\langle w, w + e_2 \rangle$ is a bond of π and $\langle w + e_1, w + e_1 + e_2 \rangle$ is a bond of ρ' . So we can perform a concatenation at the plaquette $P(w)$: let

$$\psi = (\pi \cup \rho') \Delta P(w)$$

Then $\psi \in \mathcal{Q}_{2n+16}$.

Now suppose that ψ has been constructed using the above procedure. Can we determine the original τ , σ , and $\text{proj}_1(x)$? By Lemma 3.1, we can reconstruct π and ρ' . Since there are 10 subcases of the above construction (from Ia through IIIcii), each π could have come from at most 10 different τ 's. Similarly, ρ' could have come from at most 10 different polygons $\sigma + x + K(x)e_1$. But each possible $\sigma + x + K(x)e_1$ determines the same value of $\text{proj}_1(x)$, as in the proof of Lemma 2.2. Therefore

$$\begin{aligned} \text{number of } \psi\text{'s} &\geq \frac{1}{100} \sum_{\tau, \sigma \in \mathcal{Q}_n} |\text{Proj}_1(\tau)| \\ &= \frac{q_n}{100} \frac{1}{2} \sum_{i=1}^2 \sum_{\tau \in \mathcal{Q}_n} |\text{Proj}_i(\tau)| \quad (\text{by symmetry of } \mathbf{Z}^2) \\ &\geq \frac{q_n}{100} \sum_{\tau \in \mathcal{Q}_n} n^{1/2} \quad [\text{by (10)}] \\ &= \frac{n^{1/2} q_n^2}{100} \end{aligned}$$

Since the number of ψ 's is at most q_{2n+16} , the lemma is proved (with $D = 1/100$). ■

Proof of Theorem 1.2. This follows immediately from Lemma 3.2 and Lemma 2.3 with $v_m = q_{2m}$ for $m \geq 2$. (We obtain a value of $C = 100\lambda_q^{16}/2^{1/2}$ for the constant in the statement of the theorem.) ■

4. PROOF OF THE INEQUALITY $\theta \geq v$

In this section we shall prove Theorem 1.3 for the case of trees on a homogeneous two-dimensional lattice. The proofs for animals, site animals, and self-avoiding polygons are virtually identical.

First, we shall define our terms. Let τ be a tree (or animal, etc.). For each $i = 1, \dots, d$, define

$$\begin{aligned} \text{Max}_i(\tau) &:= \max\{x_i : (x_1, \dots, x_d) \text{ is a site of } \tau\} \\ \text{Min}_i(\tau) &:= \min\{x_i : (x_1, \dots, x_d) \text{ is a site of } \tau\} \end{aligned}$$

Next, define the *mean span* of τ to be

$$MS(\tau) := \frac{1}{d} \sum_{i=1}^d [\text{Max}_i(\tau) - \text{Min}_i(\tau)]$$

Thus, the mean span of τ is the average side length of the smallest box that contains τ (and has sides parallel to the coordinate axes). Finally, we define the average mean span for trees of n sites:

$$\langle MS \rangle_n := \frac{1}{t_n} \sum_{\tau \in \mathcal{F}_n} MS(\tau) \tag{20}$$

The scaling assumption for the average mean span is that there exist constants B and ν such that

$$\langle MS \rangle_n \sim Bn^\nu \quad \text{as } n \rightarrow \infty \tag{21}$$

Now, observe that for the lattice \mathbf{Z}^2 , the set $\text{Proj}_1(\tau)$ is equal to the set of integers in the interval $[\text{Min}_2(\tau), \text{Max}_2(\tau)]$. Therefore, $|\text{Proj}_1(\tau)| = \text{Max}_2(\tau) - \text{Min}_2(\tau) + 1$. In general, for a given two-dimensional homogeneous lattice \mathcal{L} , there is a constant $K \equiv K(\mathcal{L})$ such that

$$|\text{Proj}_i(\tau)| \geq K[\text{Max}_{3-i}(\tau) - \text{Min}_{3-i}(\tau)] \tag{22}$$

for every tree τ and $i = 1, 2$. Therefore we have

$$\frac{1}{2} \sum_{i=1}^2 |\text{Proj}_i(\tau)| \geq K[MS(\tau)] \quad \text{for every tree } \tau \tag{23}$$

Proof of Theorem 1.3. From the first two lines of (15), as well as (23) and (20), we find

$$\begin{aligned} t_{2n} &\geq \sum_{\tau \in \mathcal{F}_n} \sum_{i=1}^2 |\text{Proj}_i(\tau)| t_n \\ &\geq 2t_n K \sum_{\tau \in \mathcal{F}_n} MS(\tau) \\ &= 2Kt_n^2 \langle MS \rangle_n \end{aligned}$$

The theorem now follows from the scaling assumption (21) and Lemma 2.3. ■

Remark. The argument of this section breaks down for $d \geq 3$, essentially because $\text{Proj}_i(\tau)$ is an irregular shape, quite unlike a $(d-1)$ -dimensional rectangle. Thus the analog of (22), namely $|\text{Proj}_i(\tau)| \geq K \prod_{j \neq i} [\text{Max}_j(\tau) - \text{Min}_j(\tau)]$, is false.

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